Challenge Question: Let $f : \mathbf{R}\backslash\{1\} \to \mathbf{R}$ defined by $f(x) = \dfrac{\sqrt[3]{x}-1}{x-1}$.
Find $\lim\limits_{x\to 1} f(x)$.
Hint: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

Proposition 3 (Composite functions/change of variables)**.** If $\lim_{x\to c} g(x) = k$ exists and $\lim_{u\to k} f(u)$ exists, then $\lim_{x\to c} f\circ g(x) = \lim_{u\to k} f(u)$.

$u = g(x)$

Example 2.1.9. Redo the last three examples using change of variables.

$$f(x) = \frac{\sqrt[3]{x}-1}{x-1}$$

let $u = \sqrt[3]{x}$

$u^3 = x$

$$= \frac{u-1}{u^3-1}$$

$$\begin{array}{r} u^2+u+1 \\ u-1 \,\big)\, \overline{u^3-1} \end{array}$$

$$= \frac{\cancel{(u-1)}}{\cancel{(u-1)}(u^2+u+1)}$$

$$g(u) = \frac{1}{u^2+u+1}$$

$$= g(u(x))$$

when $u^3 \neq 1$

i.e. $x \neq 1$

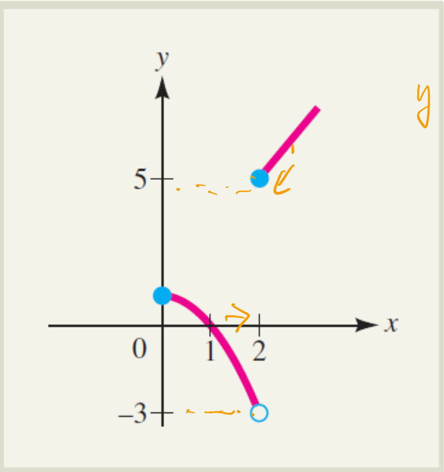
$$\lim_{x\to 1} u(x) = \lim_{x\to 1} \sqrt[3]{x} = \boxed{1} \; k$$

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1} = \lim_{u\to 1} \frac{1}{u^2+u+1} = \frac{1}{3}$$

$\square$

2.2 One-sided Limits

The following shows the graph of a piecewise function $f(x)$:



$y = f(x)$

$\lim\limits_{x \to 2} f(x)$ DNE

As x approaches 2 from the right, $f(x)$ approaches 5 and we write

$$\lim_{x \to 2^+} f(x) = 5.$$

On the other hand, as x approaches 2 from the left, $f(x)$ approaches -3 and we write

$$\lim_{x \to 2^-} f(x) = -3.$$

Limits of these forms are called one-sided limits. The limit is a right-hand limit if the approach is from the right. From the left, it is a left-hand limit.

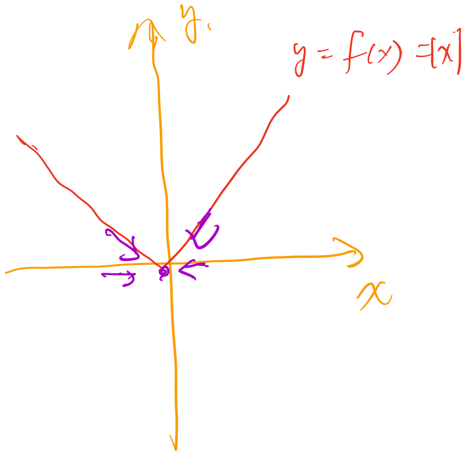
Definition 2.2.1. If $f(x)$ approaches L as x tends towards c from the left ($x < c$), we write $\lim\limits_{x \to c^-} f(x) = L$. It is called the **left-hand limit** of $f(x)$ at c.
If $f(x)$ approaches L as x tends towards c from the right ($x > c$), we write $\lim\limits_{x \to c^+} f(x) = L$. It is called the **right-hand limit** of $f(x)$ at c.

Example 2.2.1. Recall

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0.$$

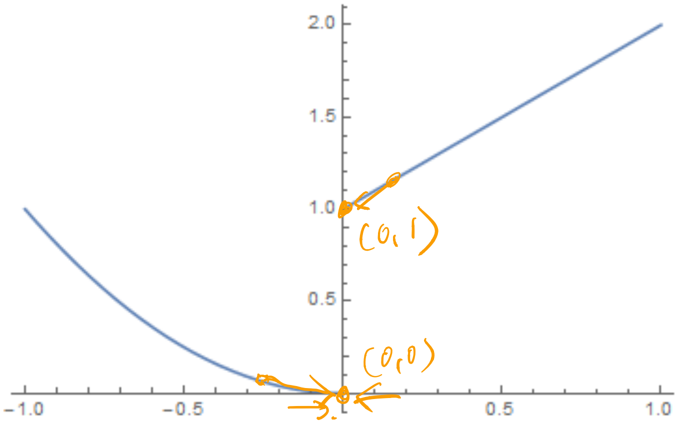
$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0.$$



For this case $\lim_{x\to 0^+}|x| = \lim_{x\to 0^-}|x|$. Then $\lim_{x\to 0}|x| = 0$.

Example 2.2.2. Define $f : \mathbf{R} \to \mathbf{R}$,

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^{-2}	10^{-4}	10^{-6}	1	1.001	1.01	1.1

We have

$$\lim_{x\to 0^+} f(x) = 1.$$

$\neq$

and

$$\lim_{x\to 0^-} f(x) = 0.$$

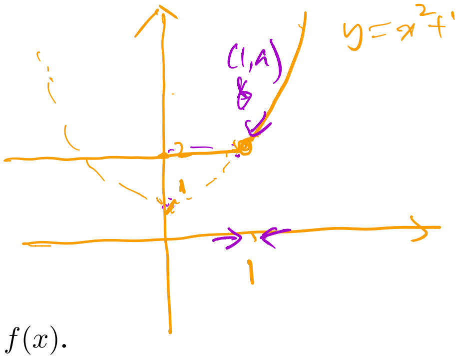
Remark.

1. The left hand limit or the right hand limit may not be the same.
2. Does $\lim_{x\to 0} f(x)$ exist? No!

Proposition 4.

$\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.

(i.e., both left hand limit and right hand limit exist and is equal to L)



Example 2.2.3. Suppose the function

$$f(x) = \begin{cases} x^2+1, & x \geq 1, \\ a, & x < 1. \end{cases}$$

has a limit as x approaches 1. Find the value of a and $\lim_{x\to 1} f(x)$.

Solution. Since $\lim_{x\to 1} f(x)$ exists, we have

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x).$$

And

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+1) = 2, \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (a) = a.$$

So, $a = 2$, and $\lim_{x\to 1} f(x) = 2$. ■

$2 = a$

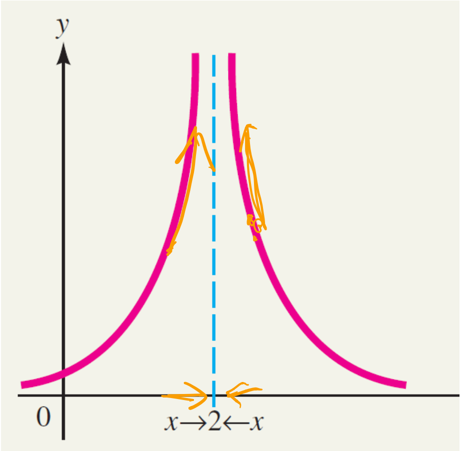
if. $\lim_{x\to 1} f(x)$ exists, $= 2$

2.3 Infinite "Limits"

Consider the following limit

$$\lim_{x\to 2} \frac{1}{(x-2)^2}.$$

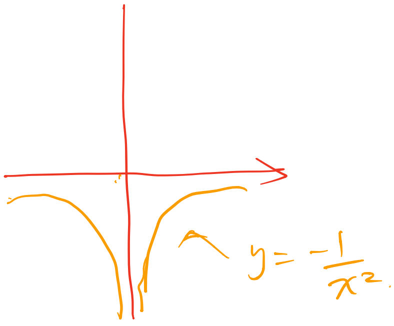
As x approaches 2, the denominator of the function $f(x) = \dfrac{1}{(x-2)^2}$ approaches 0 and hence the value of $f(x)$ becomes very large.



The function $f(x)$ increases without bound as $x \to 2$ both from left and from right. In this case, the limit *DNE (does not exist)* at $x = 2$, but we express the asymptotic behaviour of f near 2 *symbolically* as

$$\lim_{x\to 2}\frac{1}{(x-2)^2}=+\infty.$$

Remark. $+\infty$ is just a symbol, not a real number.



Example 2.3.1.

$$\lim_{x\to 0}\left(\frac{-1}{x^2}\right)=-\infty.$$

Definition 2.3.1. We say that $\lim_{x\to c} f(x)$ is an infinite limit if $f(x)$ increases or decreases without bound as $x\to c$.
If $f(x)$ increases without bound as $x\to c$, we write

$$\lim_{x\to c} f(x)=+\infty.$$

If $f(x)$ decreases without bound as $x\to c$, then

$$\lim_{x\to c} f(x)=-\infty.$$

Example 2.3.2. Evaluate

$$\lim_{x\to 2^+}\frac{x-3}{x^2-4}\quad\text{and}\quad\lim_{x\to 2^-}\frac{x-3}{x^2-4}.$$

Solution.

$$\lim_{x\to 2^+}\frac{x-3}{x^2-4}=\lim_{x\to 2^+}\frac{\overset{\to -1}{x-3}}{(x-2)\underset{}{\overset{\to 4}{(x+2)}}}=-\infty$$

positive, et smaller & small as $x \to 2^+$

since as $x\to 2^+$, we have $x^2-4=(x-2)(x+2)\to 0^+$ and $x-3\to -1^+$.

$$\lim_{x\to 2^-}\frac{x-3}{x^2-4}=\lim_{x\to 2^-}\frac{\overset{\to (-1)}{x-3}}{(x-2)(x+2)}=+\infty$$

since as $x\to 2^-$, we have $x^2-4=(x-2)(x+2)\to 0^-$ and $x-3\to -1^-$.

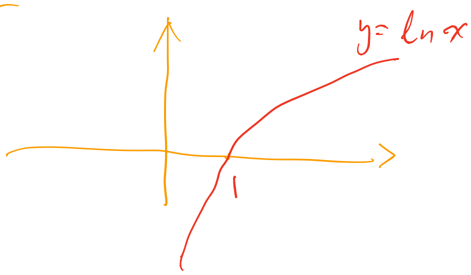
$\lim_{x\to 2}\frac{x-3}{x^2-4}$ doesn't exist.

small negative number.

■

Exercise 2.3.1. Find

$$\lim_{x\to\pi/2}\tan x;\ \overset{=+\infty}{}\qquad \lim_{x\to\pi/2^-}\tan x;\ \overset{-\infty}{}\qquad \lim_{x\to\pi/2^+}\tan x;\ \overset{DNE}{}\qquad \lim_{x\to 0^+}\ln x. = -\infty$$



Remark. Caveat! When applying the rules in Proposition 2, roughly speaking:

- “$a \pm \infty = \pm\infty$” when a is finite;
- “$\infty + \infty = \infty$”; “$-\infty - \infty - \infty$”;
- “$\infty \cdot \infty = \infty$”; “$-\infty \cdot \infty = -\infty$”; “$-\infty \cdot (-\infty) = \infty$”;
- “$a \cdot \infty = \text{sign}(a)\,\infty$” when $a \neq 0$ and is finite;
- “$\dfrac{a}{\pm\infty} = 0$” when a is finite;
- “$\dfrac{a}{0^\pm} = \pm\text{sign}(a)\,\infty$” when $a \neq 0$ and is finite;
- but “$\infty - \infty$”, “$0 \cdot \infty$”, “$\dfrac{\infty}{\infty}$”, “$\dfrac{0}{0}$” can be quite arbitrary, and must be determined case by case! We shall introduce tools to compute limits of these forms later.

← sum/difference rules in Proposition 2 extend to these cases involving infinite limits

product/quotient rules extend

$$\text{“}\frac{+\infty}{0^+}\text{”} = +\infty \qquad \text{“}\frac{\infty}{0^-}\text{”} = -\infty$$

2.4 Limits at Infinity

Definition 2.4.1. If the values of the function $f(x)$ approach the number L as x gets bigger and bigger (i.e. as x goes to $+\infty$). Then L is called the limit of $f(x)$ as x tends to $+\infty$. Denoted by

$$\lim_{x\to+\infty} f(x) = L.$$

Similarly we can define

$$\lim_{x\to-\infty} f(x) = M.$$

Remark: The value L and M may not be the same. If they are the same (i.e., $L = M$), we write

$$\lim_{x\to\infty} f(x) = L.$$

Example 2.4.1. Let $f(x) = \dfrac{1}{x}$.

-1000	-100	-10	-1	1	10	100	1000
-0.001	-0.01	-0.1	-1	1	0.1	0.01	0.001

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to+\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Proposition 5. If A and $k > 0$ are constants. Then

$$\boxed{\lim_{x\to+\infty} \frac{A}{x^k} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{A}{x^k} = 0.}$$

"$\frac{A}{\infty}$"

"$\frac{A}{-\infty}$"

when x^k is defined

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator.

Example 2.4.2. Find $\displaystyle\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1}$ $\overset{?}{=}$ $\dfrac{\lim\limits_{x\to\infty} 3x^2}{\lim\limits_{x\to\infty}(x^2+x+1)}$ = "$\dfrac{+\infty}{+\infty}$"

quotient rule

quotient rule doesn't apply

Solution.

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1} \quad \text{(Divide both the top and bottom by } x^2\text{)}$$

$$= \lim_{x\to+\infty} \frac{3}{1+\frac{1}{x}+\frac{1}{x^2}}$$

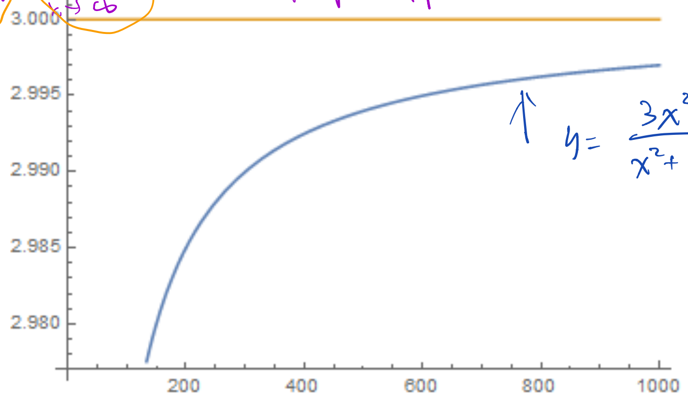
from prop 5.

$$= \frac{\lim\limits_{x\to\infty} 3}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \frac{1}{x} + \lim\limits_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{3}{1+0+0} = 3.$$

the algebraic rules in prop2 applies

$\lim\limits_{x\to\infty} \frac{1}{x} = 0 = \lim\limits_{x\to\infty} \frac{1}{x^2}$



$y = \dfrac{3x^2}{x^2+x+1}$

■

Question: Can we write

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1} = \frac{\lim\limits_{x\to+\infty} (3x^2)}{\lim\limits_{x\to+\infty} (x^2+x+1)}?$$

Hint: Recall the Caveat from the end of last section.

Example 2.4.3. Find $\displaystyle\lim_{x\to+\infty} \frac{x-1}{2x^2+3x+1}$

~ algebraic rules from prop 2

"$\dfrac{\infty}{\infty}$" ← exceptional case when prop 2 doesn't apply

Solution.

$$\lim_{x\to+\infty}\frac{x-1}{2x^2+3x+1} \quad \text{(Divide both the top and bottom by } x^2\text{)}$$

$$=\lim_{x\to+\infty}\left(\frac{\frac{1}{x}-\frac{1}{x^2}}{2+3\frac{1}{x}+\frac{1}{x^2}}\right)$$

$$=\frac{0}{2+0+0}=0.$$

Prop 2 can be applies to this formula

$$\frac{\lim_{x\to\infty}\frac{1}{x}-\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}2+\lim_{x\to\infty}\frac{3}{x}+\lim_{x\to\infty}\frac{1}{x^2}}$$

$$" \frac{0-0}{2+0+0}$$

■

Example 2.4.4. Find $\lim_{x\to+\infty}\frac{x^3-1}{2x^2+3x+1}$.

Solution.

$$\lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{\frac{2}{x}+\frac{3}{x^2}+\frac{1}{x^3}} \longrightarrow$$

~ prop2

$$\frac{}{\text{" }0\text{ "}}$$

$$\lim_{x\to+\infty}\frac{x^3-1}{2x^2+3x+1}$$

$$=\lim_{x\to+\infty}\left(\frac{x-\frac{1}{x^2}}{2+\frac{3}{x}+\frac{1}{x^2}}\right)=$$

$$=+\infty.$$

$$\frac{\overset{+\infty}{\lim_{x\to\infty}x}-\overset{0}{\lim_{x\to\infty}\frac{1}{x^2}}}{2+0+0}$$

$$= \text{"}\frac{\infty}{2}\text{"} = +\infty$$

■

Proposition 6. Suppose

when $x\to\infty$, the leading term would be much larger than the rest of the terms

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0,\ a_n\neq 0$$

$$q(x)=b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0,\ b_m\neq 0$$

ignorable

Then

$$\lim_{x\to+\infty}\frac{p(x)}{q(x)}=\begin{cases}\frac{a_n}{b_m} & \text{if } n=m, \text{ when } x\to\infty\\ 0 & \text{if } n<m,\\ +\infty & \text{if } a_nb_m>0,\ n>m,\\ -\infty & \text{if } a_nb_m<0,\ n>m.\end{cases}$$

$$= \lim_{x\to\infty}\left(\frac{a_nx^n}{b_mx^m}\right)$$

when $n>m$

$$\lim_{x\to\infty}\frac{x^n}{x^m}=\lim_{x\to\infty}x^{n-m}$$

$$=+\infty$$

Remark. One way to see this: The leading term in a polynomial dominates the lower order terms as $x\to\pm\infty$. (Higher powers of x "grows faster" than lower powers of x as $x\to\infty$. Log functions grows slower than any polynomial function because (as we'll see later) $\lim_{x\to\infty}\frac{\ln x}{x^a}=0$ for any $a>0$.

$$\lim_{x\to\infty}\log_a x=+\infty$$

L'Hôpital

Example 2.4.5. Find $\lim_{x\to\infty}\frac{3x^3-2x^2+1}{-x^3+7}$.

$$=\lim_{x\to\infty}\left(\frac{3x^3}{-x^3}\right)=\lim_{x\to\infty}(-3)=-3$$

Solution. By the proposition, the answer is $\frac{3}{-1}=-3$. ■

Similar technique can be used for functions with radical (i.e., something like $\sqrt{x}$).

Example 2.4.6. Find $\lim_{x\to+\infty} \frac{3x-1}{\sqrt{3x^2+1}}$. $= \text{"}\frac{\infty}{\infty}\text{"}$

↑ Prop 2 doesn't apply

Solution. The term with highest degree of the denominator is x^2. But we need to take square root. So we divide the nominator and the denominator by $\sqrt{x^2} = x$. We have

$$\lim_{x\to\infty} \frac{3x}{\sqrt{3x^2}} = \lim_{x\to+\infty} \left(\frac{3x-1}{\sqrt{3x^2+1}}\right) = \lim_{x\to+\infty} \frac{\frac{1}{x}(3x-1)}{\frac{1}{x}\sqrt{3x^2+1}}$$

$$= \lim_{x\to\infty} \sqrt{3} \qquad = \lim_{x\to+\infty} \frac{3-\frac{1}{x}}{\underbrace{\sqrt{3+\frac{1}{x^2}}}_{}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

↑ apply Prop 2.

■

Example 2.4.7. **(Rationalization)**
Evaluate

$$\lim_{x\to+\infty} (\sqrt{x+1} - \sqrt{x}). \qquad \text{"}\infty - \infty\text{"}$$

exceptional case.
Prop 2 doesn't directly apply

Solution. (Recall the *Caveat* from last section!)

$$\lim_{x\to+\infty} (\sqrt{x+1} - \sqrt{x}) = \lim_{x\to+\infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} = \frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}}$$

$$(a-b)(a+b) = a^2 - b^2$$

$$= \lim_{x\to+\infty} \boxed{\frac{1}{\sqrt{x+1} + \sqrt{x}}}$$

$$= 0.$$

↑ apply Prop 2.

$$= \frac{1}{\text{"}\infty + \infty\text{"}} = 0$$

■

Exercise 2.4.1.

1. $\lim_{x\to-\infty} \frac{x^3+1}{-2x^3+x} = -\frac{1}{2}$.

2. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$ (Caution: $x < 0 \Rightarrow \frac{1}{x} = -\sqrt{\frac{1}{x^2}}$).

3. $\lim_{x\to+\infty} (\sqrt{x^2+x} - \sqrt{x^2-2}) = \frac{1}{2}$.

Example 2.4.8. $\lim_{x\to+\infty} \sin x = ?$ DNE